

Recall: $\varphi: G \rightarrow G'$ is said to be a homom.

if $\varphi(ab) = \varphi(a)\varphi(b) \quad \forall a, b \in G.$

$\varphi(1) = 1$
 $\varphi(a^{-1}) = \varphi(a)^{-1}$

Defⁿ $\text{Im}(\varphi) = \{x' \in G' : x' = \varphi(x) \text{ for some } x \in G\}$
 (Range of φ)

Propⁿ

$\text{Im}(\varphi) \leq G'$ is a subgp.

Pf: Need to check: $\text{Im}(\varphi)$ is closed under multiplication

i.e. if $x \in \text{Im}(\varphi), y \in \text{Im}(\varphi)$ then $xy \in \text{Im}(\varphi).$

$\exists a \quad x = \varphi(a) \quad \exists b \text{ s.t. } y = \varphi(b)$

$xy = \varphi(a)\varphi(b) = \varphi(ab) \quad \checkmark \quad \text{QED}$

$\varphi: G \rightarrow G'$ a homom.

Defⁿ $\text{Ker}(\varphi)$, the kernel of φ , is defined as

$\text{Ker}(\varphi) = \{g \in G : \varphi(g) = 1 \in G'\}$

Propⁿ $\text{Ker}(\varphi)$ is a subgp of G .

Pf: clearly $1 \in \text{Ker}(\varphi)$ b/c $\varphi(1) = 1$

• If $a \in \text{Ker}(\varphi)$ then $\varphi(a^{-1}) = \varphi(a)^{-1} = 1^{-1} = 1$
 $\Rightarrow a^{-1} \in \text{Ker}(\varphi)$

$$\Rightarrow a \in \ker(\varphi) \text{ then } \varphi(a)^{-1} = \varphi(a)^{-1} = 1 = 1 \\ \Rightarrow a^{-1} \in \ker(\varphi)$$

• If $a, b \in \ker(\varphi)$ then

$$\varphi(ab) = \varphi(a)\varphi(b) = 1 \cdot 1 = 1 \\ \text{hence } a, b \in \ker(\varphi)$$

$$\Rightarrow ab \in \ker(\varphi).$$

QED

Examples

• $\text{sgn}: S_n \rightarrow \{\pm 1\}$

We defined $A_n \subseteq S_n$ as $\ker(\text{sgn})$

• $\det: GL_n(\mathbb{R}) \rightarrow \mathbb{R}^\times$ has kernel $SL_n(\mathbb{R})$.

Propⁿ Suppose $\varphi: G \rightarrow G'$ is a homom. $K = \ker(\varphi)$

IFAB

① $\varphi(a) = \varphi(b)$

② $a^{-1}b \in K$

③ $\exists k \in K$ s.t. $b = ak$.

Pf. We'll check ① $\Rightarrow$ ② $\Rightarrow$ ③

① $\Rightarrow$ ② $\varphi(a) = \varphi(b) \Rightarrow \varphi(a)^{-1} \varphi(b) = 1$

$\Rightarrow \varphi(a^{-1}) \varphi(b) = 1 \Rightarrow \varphi(a^{-1}b) = 1$

$\Rightarrow a^{-1}b \in K.$

$$\Rightarrow a^{-1}b \in K.$$

$$\underline{(2) \Rightarrow (3)} \quad a^{-1}b \in K \text{ means } \exists k \in K \text{ s.t. } a^{-1}b = k \\ \Rightarrow b = ak.$$

$$\underline{(3) \Rightarrow (1)} \quad b = ak \text{ for some } k \in K. \\ \Rightarrow \varphi(b) = \varphi(ak) = \varphi(a)\varphi(k) = \varphi(a) \cdot 1 = \varphi(a). \quad \square$$

Cor $\varphi: G \rightarrow G'$ is a homom. Then φ is injective

$$\iff K = \text{Ker}(\varphi) = \{1\}.$$

$$\underline{\text{Pf:}} \quad \varphi \text{ injective} \Rightarrow \left\{ \varphi(k) = \varphi(1) \Rightarrow k = 1 \right\} \\ k \in G \quad \quad \quad = 1$$

$$\varphi(k) = 1 \Rightarrow k = 1$$

$$\Rightarrow \text{Ker}(\varphi) = 1.$$

• Suppose $\text{Ker}(\varphi) = \{1\}$, $a, b \in G$ w/ $\varphi(a) = \varphi(b)$.

$$\Rightarrow a^{-1}b \in \text{Ker}(\varphi) = \{1\}$$

$$a^{-1}b = 1 \Rightarrow a = b. \quad \square$$

Defⁿ If $a, g \in G$, then gag^{-1} is called the conjugate of a by g .

Defⁿ A subgp N of G is said to be normal if $\forall a \in N$ and $g \in G$, the conjugate $gag^{-1} \in N$.

$\forall u \in N$ and $g \in G$, the conjugate $gag^{-1} \in N$.

Propⁿ The kernel of a homom. is normal.

Pf: Suppose $\varphi: G \rightarrow G'$, and $a \in \ker(\varphi)$.

Take $g \in G$. Then

$$\begin{aligned}\varphi(gag^{-1}) &= \varphi(g) \varphi(a) \varphi(g^{-1}) \\ &= \varphi(g) \cdot 1 \cdot \varphi(g)^{-1} \\ &= 1 \quad \square\end{aligned}$$

Later We'll check the converse, i.e. if $N \leq G$ is a normal subgroup then $\exists$ a gp G' and a homom $\varphi: G \rightarrow G'$ with $\ker(\varphi) = N$.

Ex • $A_n \leq S_n$ is normal

• $SL_n(\mathbb{R}) \leq GL_n(\mathbb{R})$ is also normal

- If G is an abelian gp, and $H \leq G$ is a subgroup, then H is normal.

Pf: Suppose $h \in H$, $g \in G$. Then

$$ghg^{-1} = hgg^{-1} = h \cdot 1 = h \in H.$$

- $S_3 \ni \{1, x, x^2\}$ is normal

14 of 14

First note that $\lambda \cdot I_n$, if $\lambda \in \mathbb{R}^x$, is in the center of $GL_n(\mathbb{R})$.

$$\text{I.e. } (\lambda I_n)g = g(\lambda I_n) \quad \forall g \in GL_n(\mathbb{R}).$$

Opposite direction:

Suppose $m = (m_{ij}) \in GL_n(\mathbb{R})$, and suppose m commutes w/ every matrix of the form

$$z = \begin{pmatrix} z_1 & & 0 \\ & z_2 & \\ 0 & & z_n \end{pmatrix} \quad \text{w/ } z_j \in \mathbb{R}^x \text{ for all } j.$$

$$mz = zm \implies zmz^{-1} = m.$$

$$zmz^{-1} = \left(\frac{z_i}{z_j} m_{ij} \right) = (m_{ij})$$

So if z commutes w/ m then

$$\frac{z_i}{z_j} m_{ij} = m_{ij} \quad \forall i, j.$$

Choose z so that all z_i 's are distinct. Then

$$\frac{z_i}{z_j} \neq 1 \quad \text{if } i \neq j.$$

$$\text{Thus if } i \neq j \quad \frac{z_i}{z_j} m_{ij} = m_{ij} \implies m_{ij} = 0.$$

$\Rightarrow m$ is diagonal, $m = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$ for some $\lambda_j \in \mathbb{R}^X$.

To see that $\lambda_1 = \lambda_2 = \dots = \lambda_n$,
consider $L(p)$ for $p \in S_n$.

Suppose $\underline{L(p)m = m L(p)}$

Apply both sides to $e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix}$ a 1 in the j^{th} position

- $\bullet L(p)m(e_j) = L(p)(\lambda_j e_j) = \lambda_j L(p)(e_j) = \lambda_j e_{p(j)}$
- $\bullet m L(p)(e_j) = m(e_{p(j)}) = \lambda_{p(j)} e_{p(j)}$

Have: $\lambda_{p(j)} e_{p(j)} = \lambda_j e_{p(j)} \quad \forall j \text{ and all } p \in S_n.$

Take $p \text{ w/ } p(j) \neq j \Rightarrow \lambda_{p(j)} = \lambda_j.$

$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n$, as desired. $\square$

Example Center of S_n is $\{1\}$ if $n \geq 3$.

Pf: HW.

Isomorphisms

An iso $\varphi: G \rightarrow G'$ is a bijective homom.

Propⁿ Suppose $\varphi: G \rightarrow G'$ is an isom.

Let $\varphi^{-1}: G' \rightarrow G$ be the inverse map

Then φ^{-1} is an isom, i.e. φ^{-1} is a bijective homom.

Pf:

NTP: $\forall x, y \in G'$ that

$$\star \varphi^{-1}(xy) = \varphi^{-1}(x) \varphi^{-1}(y).$$

Because φ is a bijection, it is in particular injective, and thus to check $(\star)$ it suffices to apply φ , i.e. to check

$$\varphi(\varphi^{-1}(xy)) = \varphi(\varphi^{-1}(x) \varphi^{-1}(y)).$$

But: LHS = xy

RHS = $\varphi(\varphi^{-1}(x)) \varphi(\varphi^{-1}(y))$ b/c φ is a homom.

$$= x \cdot y$$

$$= xy \checkmark.$$

□

Defⁿ G is said to be isom. to G' if $\exists \varphi: G \rightarrow G'$ an isom.

Examples

① $\exp: (\mathbb{R}, +) \rightarrow \mathbb{R}_{>0}^{\times}$
 $x \mapsto e^x$

Is an isom.

② Suppose $g \in G$. Define $\varphi_g: G \rightarrow G$ as

$$\varphi_g(x) = gxg^{-1}.$$

Then φ_g is an isom.

Pf: φ_g is a homom b/c

$$\begin{aligned}\varphi_g(xy) &= gxyg^{-1} = gxg^{-1}gyg^{-1} \\ &= (gxg^{-1})(gyg^{-1}) = \varphi_g(x)\varphi_g(y)\end{aligned}$$

Need to check: φ_g is bijective.

This follows from

Lemma: G gp, $a \in G$ an elt.

Define $L_a: G \rightarrow G$ as

$$L_a(x) = ax,$$

Similarly, define

$R_a: G \rightarrow G$ as

$$R_a(x) = xa.$$

Then: L_a and R_a are bijections.

Lemma implies Propⁿ Note that $\varphi_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g$
and the composition of bijections is a
bijection

Pf of Lemma

$L_a: G \rightarrow G$ is

Proposition:

$L_a: G \rightarrow G$ is

surjective: Take $y \in G$. Then $L_a(a^{-1}y) = aa^{-1}y = y$

so y is in range of L_a .

injective: Suppose $x, y \in G$ and $L_a(x) = L_a(y)$.

$$\Rightarrow ax = ay \Rightarrow x = y.$$

Case of R_a is completely analogous. $\square$